

Second Moment Boundedness of Linear Stochastic Delay Differential Equations

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Abstract

This paper studies the second moment boundedness of solutions of linear stochastic delay differential equations. First, we give a framework, for general N -dimensional linear stochastic differential equations with a single discrete delay, of calculating the characteristic function for the second moment boundedness. Next, we apply the proposed framework to a special case of a type of 2-dimensional equation that the stochastic terms are decoupled. For the 2-dimensional equation, we obtain the characteristic function explicitly given by equation coefficients, the characteristic function gives sufficient conditions for the second moment to be bounded or unbounded.

Keywords: Ito integral, Laplace transform, stochastic differential equation, characteristic function

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1. Introduction

Stochastic delay differential equations have been extensively studied in the last several decades from different points of view (see [3, 4, 6–13, 15, 16] and the references therein). However, many basic issues remain unsolved even for linear equations with constant coefficients.

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In this paper, we study general N -dimensional linear stochastic delay differential equations with a single discrete delay (here we assume the delay $\tau = 1$)

$$dx_i(t) = \left[a_i^j x_j(t) + b_i^j x_j(t-1) \right] dt + \left[\mu_i^k + \sigma_i^{jk} x_j(t) + \eta_i^{jk} x_j(t-1) \right] dW_k. \\ (i, j = 1, \dots, N; \quad k = 1, \dots, K) \quad (1.1)$$

Here the *Einstein summation convention* has been used so that repeated indices are implicitly summed over, all coefficients a_i^j , b_i^j , μ_i^k , σ_i^{jk} , η_i^{jk} are constants, and W_k are independent 1-dimensional Wiener Processes. The initial functions are assumed to be $x_i = \phi_i \in C([-1, 0], \mathbb{R})$ ($i = 1, \dots, N$). This paper considers the second moment boundedness of solutions of (1.1). We always assume Itô interpretation for the stochastic integral, and results for Stratonovich interpretation can be obtained similarly.

To the best of our knowledge, there are very few results for the stability and second moment boundedness for equation (1.1), and most of known results are obtained through the method of Lyapunov functional [9–11, 14]. However, it remains unclear how can we define the characteristic equation for the boundedness of the solution moments of (1.1). In 2007, Lei and Mackey [8] introduced the method of Laplace transform to study the second moment boundedness of 1-dimensional equations with a single discrete delay, and proposed a characteristic equation. In [17], the authors extended the method to the 1-dimensional equation with distributed delay.

Based on previous studies [8, 17], this paper aims at proposing a general framework to calculate the characteristic function for high dimensional linear stochastic delay differential equations with a single delay. The obtained characteristic function (as we can see below) is complicate, that shows the elaborate correlations when delay and stochastic effects are coupled into an dynamical system. As an example, we apply the framework to study a special situation of a 2-dimensional equation in which the stochastic terms are decoupled ($N = K = 2$, and $\mu_i^k = \sigma_i^{jk} = \eta_i^{jk} = 0$ when $i \neq k$).

Rest of this paper is organized as follows. In Section 2, we briefly introduce basic results for linear delay differential equations. In Section 3, a general framework for defining the characteristic function of a N -dimensional stochastic delay differential equation is given. In Section 4, we discuss the boundedness of the second moment of (1.1) for a simple case: $N = K = 2$ and the stochastic terms are decoupled. Theorem 4.1 establishes the unbounded condition for the second moment if the trivial solution of the unperturbed

equation is unstable. When the trivial solution of the unperturbed equation is stable, we obtain a characteristic function given explicitly by the equation parameters. Boundedness of the second moments depends on the supremum of the real parts of all roots of the characteristic equation (Theorem 4.3). An explicit condition for the boundedness of the second moment is also proved following framework of calculation given here (Theorem 4.7).

2. Preliminaries

In this paper, we always use the L^1 norm for a tensor. For example, the L^1 norm of a second order tensor $A = \{a_i^j\}$ is

$$\|A\| = \sum_{i,j=1}^N |a_i^j|. \quad (2.1)$$

For $\phi = (\phi_i)_{N \times 1} \in C([-1, 0], \mathbb{R}^N)$, the norm is defined as

$$\|\phi\| = \sup_{\theta \in [-1, 0]} \sum_{i=1}^N |\phi_i(\theta)|. \quad (2.2)$$

In this section we give some basic results for the N -dimensional linear delay differential equation

$$\frac{dx_i(t)}{dt} = a_i^j x_j(t) + b_i^j x_j(t-1) \quad (i, j = 1, \dots, N) \quad (2.3)$$

with initial functions $x_i = \phi_i \in C([-1, 0], \mathbb{R})$. The linear autonomous functional differential equation (2.3) has been studied extensively and details can be referred to [1, 2, 5].

The fundamental matrix of (2.3), denoted by

$$X(t) = (X_i^j(t))_{N \times N},$$

is the solution of (2.3) with the initial condition (hereinafter δ means the Kronecker delta)

$$X_i^j(t) = \begin{cases} \delta_i^j, & t = 0, \\ 0, & -1 \leq t < 0. \end{cases}$$

Using the fundamental matrix $X(t)$, the solution of (2.3) with initial function $\phi \in C([-1, 0], \mathbb{R}^N)$ can be represented as

$$x_{\phi,i}(t) = X_i^j(t)\phi_j(0) + \int_{-1}^0 X_i^j(t-1-\theta)b_j^l\phi_l(\theta)d\theta. \quad (2.4)$$

From (2.4), the asymptotic behavior of $x_\phi(t) = (x_{\phi,i}(t))_{N \times 1}$ is determined by the fundamental matrix $X(t)$.

Denote the matrices

$$A = (a_i^j)_{N \times N}, \quad B = (b_i^j)_{N \times N}, \quad I = (\delta_i^j)_{N \times N}, \quad \mu = (\mu_i^j)_{N \times N}. \quad (2.5)$$

Taking the Laplace transform on both sides of (2.3), we obtain

$$\mathcal{L}(X)(\lambda) = [\Delta(\lambda)]^{-1}, \quad (2.6)$$

where

$$\Delta(\lambda) = \lambda I - A - B e^{-\lambda}. \quad (2.7)$$

Thus, $h(\lambda) = \det(\Delta(\lambda))$ is the characteristic function for the linear stability of (2.3).

The following results are straightforward from the above discussions.

Theorem 2.1. *Let*

$$\alpha_0 = \sup\{\operatorname{Re}(\lambda) : h(\lambda) = 0, \lambda \in \mathbb{C}\}. \quad (2.8)$$

Then

- (i) *for any $\alpha > \alpha_0$ there exists a constant $\bar{K} = \bar{K}(\alpha) \geq 1$ such that the fundamental matrix $X(t)$ satisfies*

$$\|X(t)\| \leq \bar{K} e^{\alpha t}, \quad t \geq 0;$$

- (ii) *for any $\alpha > \alpha_0$ there exists a constant $\tilde{K} = \tilde{K}(\alpha) \geq 1$ such that for any $\phi \in C([-1, 0], \mathbb{R}^N)$ the solution $x_\phi(t)$ of (2.3) satisfies*

$$\|x_\phi(t)\| \leq \tilde{K} \|\phi\| e^{\alpha t}, \quad t \geq 0;$$

- (iii) *for any $\alpha_1 < \alpha_0$, there exists $\bar{\alpha} \in (\alpha_1, \alpha_0)$ and a subset $U \subset \mathbb{R}^+$ with measure $m(U) = +\infty$ such that for any $i, j = 1, \dots, N$,*

$$\|X_i^j(t)\| \geq e^{\bar{\alpha} t}, \quad t \in U. \quad (2.9)$$

Here, the number α_0 is also termed as the *Lyapunov exponent* (Definition 1.19 in [1]). The proofs of (i) and (ii) in Theorem 2.1 is referred to the proofs in [1, Theorem 1.21 in Chapter 3]. The proof of (iii) is similar to that of [17, Theorem 2.3, 2] and is detailed at Appendix A.

From Theorem 2.1, the trivial solution of (2.3) is locally asymptotically stable if and only if $\alpha_0 < 0$.

3. Moments of linear stochastic delay differential equations—General cases

Now, we discuss the solution moments and a framework for calculating the characteristic function of the second moment of solutions for general cases.

The existence and uniqueness results for stochastic delay differential equations have been established in [6, 11, 14]. Using the fundamental matrix $X(t)$, the solution of (1.1) with initial function $x = \phi \in C([-1, 0], \mathbb{R}^N)$ is a N -dimensional stochastic process given by Itô integral as follows

$$x_i(t; \phi) = x_{\phi,i}(t) + \int_0^t X_i^l(t-s) (\mu_l^k + \sigma_l^{jk} x_j(s; \phi) + \eta_l^{jk} x_j(s-1; \phi)) dW_k, \quad t \geq 0, \quad (3.1)$$

where $(x_{\phi,i}(t))_{N \times 1}$ is the solution of (2.3) defined by (2.4).

We denote by E the mathematical expectation. Now we give the definitions of the p^{th} moment exponential stability and the p^{th} moment boundedness.

Definition 3.1. A solution $x(t; \phi)$ of (1.1) is said to be the first moment exponentially stable if there exist two positive constants γ and R such that

$$\|E(x(t; \phi))\| \leq R\|\phi\|e^{-\gamma t}, \quad t \geq 0,$$

for all $\phi \in C((-1, 0], \mathbb{R}^N)$. When $p \geq 2$, a solution of (1.1) is said to be the p^{th} moment exponentially stable if there exist two positive constants γ and R such that

$$E(\|x(t; \phi) - E(x(t; \phi))\|^p) \leq R\|\phi\|^p e^{-\gamma t}, \quad t \geq 0,$$

for all $\phi \in C((-1, 0], \mathbb{R}^N)$.

Definition 3.2. For $p \geq 2$, a solution $x(t; \phi)$ of (1.1) is said to be the p^{th} moment bounded if there exists a positive constant $\tilde{C} = \tilde{C}(\|\phi\|^p)$ such that

$$E(\|x(t; \phi) - E(x_i(t; \phi))\|^p) \leq \tilde{C}, \quad t \geq 0,$$

for all $\phi \in C((-1, 0], \mathbb{R}^N)$. Otherwise, the p^{th} moment is said to be unbounded.

3.1. First moment

From (3.1) and applying the properties of Itô integral, it is easy to have

$$Ex_i(t; \phi) = x_{\phi, i}(t) \quad (i = 1, \dots, N).$$

Thus, the following result is straightforward from Theorem 2.1.

Theorem 3.3. Let α_0 be defined by (2.8). For any $\alpha > \alpha_0$, there exists a positive constant \tilde{K} (defined as Theorem 2.1) such that for any $\phi \in C([-1, 0], \mathbb{R}^N)$, the solution $x(t; \phi) = (x_i(t; \phi))_{N \times 1}$ of (1.1) satisfies

$$\|Ex(t; \phi)\| \leq \tilde{K}\|\phi\|e^{\alpha t}, \quad t \geq 0. \quad (3.2)$$

In particular, the first moment of (1.1) is exponentially stable when $\alpha_0 < 0$.

3.2. Second moment

Now we study the second moment. First, we give some notations. Let $x(t; \phi) \triangleq (x_i(t))_{N \times 1}$ be a solution of (1.1) and $\tilde{x}_i(t) = x_i(t) - E(x_i(t))$ ($i = 1, \dots, N$), and define

$$M_{ij}(t) = E(\tilde{x}_i(t)\tilde{x}_j(t)), \quad N_{ij}(t) = E(\tilde{x}_i(t)\tilde{x}_j(t-1)). \quad (3.3)$$

Then $M_{ii}(t)$ is the second moment of $x_i(t)$. It is easy to have $\tilde{x}_i(t) = M_{ij}(t) = N_{ij}(t) = 0$ when $t \in [-1, 0]$, and $E(\tilde{x}_i(t)) = 0$ for all $t \geq 0$.

Denote

$$\Sigma_l^k(s) = \mu_l^k + \sigma_l^{jk} x_j(s) + \eta_l^{jk} x_j(s-1). \quad (3.4)$$

From (3.1), we have

$$\tilde{x}_i(t) = \int_0^t X_i^l(t-s) \Sigma_l^k(s) dW_k.$$

Since $E(dW_k dW_m) = \delta_{k,m}$, we obtain

$$M_{ij}(t) = E(\tilde{x}_i(t)\tilde{x}_j(t)) = \int_0^t X_i^l(t-s) E[\Sigma_l^k(s) \delta_{k,m} \Sigma_p^m(s)] X_j^p(t-s) ds. \quad (3.5)$$

3.2.1. Additive noise

We have additive noise when $\sigma_l^{jk} = \eta_l^{jk} = 0$ for any i, j, k and $\mu \neq 0$ (μ is defined at (2.5)). In this case, we have

$$M_{ij}(t) = \int_0^t X_i^l(s) X_j^p(s) \mu_l^k \delta_{k,m} \mu_p^m ds. \quad (3.6)$$

Thus the upper bound of $M(t)$ is determined by that of the fundamental matrix $X(t)$. Hence, we have the following sufficient conditions for the second moments $M(t)$ to be bounded or unbounded.

Theorem 3.4. *Let α_0 be defined as (2.8). Assume $\sigma_l^{jk} = \eta_l^{jk} = 0$ for any i, j, k and $\mu \neq 0$. Then the second moment of (1.1) is bounded if $\alpha_0 < 0$, and unbounded if $\alpha_0 > 0$.*

The proof is similar to that in [17, Theorem 3.4] and is omitted here.

The critical case $\alpha_0 = 0$ is not discussed here and the boundedness issue remains open.

3.2.2. A general framework

Now, we consider the general situation. Let

$$\begin{aligned} P_{lp}(s) &= E(\Sigma_l^k(s)) \delta_{k,m} E(\Sigma_p^m(s)), \\ Q_{lp}(s) &= E \left[\left(\sigma_l^{jk} \tilde{x}_j(s) + \eta_l^{jk} \tilde{x}_j(s-1) \right) \delta_{k,m} \left(\sigma_p^{qm} \tilde{x}_q(s) + \eta_p^{qm} \tilde{x}_q(s-1) \right) \right]. \end{aligned}$$

Then $P_{lp}(s) = P_{pl}(s)$, $Q_{lp}(s) = Q_{pl}(s)$ and

$$\begin{aligned} Q_{lp}(s) &= \sigma_l^{jk} \delta_{k,m} \sigma_p^{qm} M_{jq}(s) + \eta_l^{jk} \delta_{k,m} \eta_p^{qm} M_{jq}(s-1) \\ &\quad + \left(\sigma_l^{jk} \delta_{k,m} \eta_p^{qm} + \eta_l^{jk} \delta_{k,m} \sigma_p^{qm} \right) N_{jq}(s). \end{aligned} \quad (3.7)$$

We note that $x_i(s) = \tilde{x}_i(s) + E x_i(s)$, then

$$E \left[\Sigma_l^k(s) \delta_{k,m} \Sigma_p^m(s) \right] = P_{lp}(s) + Q_{lp}(s).$$

Therefore,

$$M_{ij}(t) = \int_0^t X_i^l(t-s) X_j^p(t-s) (P_{lp}(s) + Q_{lp}(s)) ds.$$

Denote

$$F_{ij}(t) = \int_0^t X_i^l(t-s)X_j^p(t-s)P_{lp}(s)ds, \quad (3.8)$$

then

$$M_{ij}(t) = F_{ij}(t) + \int_0^t X_i^l(t-s)X_j^p(t-s)Q_{lp}(s)ds. \quad (3.9)$$

Similarly, we have

$$N_{ij}(t) = \int_0^{t-1} X_i^l(t-s)X_j^p(t-1-s)(P_{lp}(s) + Q_{lp}(s))ds. \quad (3.10)$$

From (3.9), we have $|M_{ii}(t)| \geq |F_{ii}(t)|$ ($\forall i$). If $\alpha_0 > 0$ (α_0 is defined by (2.8)), similar to discussions in [17], $|F_{ii}(t)|$ approaches to infinity exponentially, and therefore the second moment is unbounded (a proof for the case of a 2-dimensional equation is given in the next section). Thus, we only need to study the situation when $\alpha_0 < 0$.

From (3.9) and (3.10), and take Laplacians to both sides of them (existence of the Laplacians are proved in Lemmas 3.5 and 3.6 below), we obtain

$$\mathcal{L}(M_{ij}) = \mathcal{L}(X_i^h X_j^p) [\mathcal{L}(P_{hp}) + \mathcal{L}(Q_{hp})], \quad (3.11)$$

and

$$\mathcal{L}(N_{ij}) = \mathcal{L}(X_i^h(t)X_j^p(t-1)) [\mathcal{L}(P_{hp}) + \mathcal{L}(Q_{hp})]. \quad (3.12)$$

From (3.7), we have

$$\begin{aligned} \mathcal{L}(Q_{hp}) &= (\sigma_h^{jk} \delta_{k,m} \sigma_p^{qm} + e^{-\lambda} \eta_h^{jk} \delta_{k,m} \eta_p^{qm}) \mathcal{L}(M_{jq}) \\ &\quad + (\sigma_h^{jk} \delta_{k,m} \eta_p^{qm} + \eta_h^{qk} \delta_{k,m} \sigma_p^{jm}) \mathcal{L}(N_{jq}). \end{aligned} \quad (3.13)$$

Now, one can solve $\mathcal{L}(M_{ij})$ from (3.11)-(3.13) following the procedure below. First, solve $\mathcal{L}(P_{hp}) + \mathcal{L}(Q_{hp})$ by $\mathcal{L}(M_{ij})$ from (3.11) as

$$\mathcal{L}(P_{hp}) + \mathcal{L}(Q_{hp}) = S_{hp}^{ij} \mathcal{L}(M_{ij}).$$

Here S_{hp}^{ij} is the inverse tensor of $\mathcal{L}(X_i^h X_j^p)$. Next, substitute the obtained $\mathcal{L}(P_{hp}) + \mathcal{L}(Q_{hp})$ into (3.12) to linearly express $\mathcal{L}(N_{ij})$ through $\mathcal{L}(M_{ij})$:

$$\mathcal{L}(N_{ij}) = \mathcal{L}(X_i^h(t)X_j^p(t-1))S_{hp}^{kl} \mathcal{L}(M_{kl}).$$

Then, put the resulting $\mathcal{L}(N_{ij})$ into (3.13) so that $\mathcal{L}(Q_{hp})$ linearly depends on $\mathcal{L}(M_{ij})$ in the form

$$\mathcal{L}(Q_{hp}) = T_{hp}^{kl} \mathcal{L}(M_{kl}).$$

Finally, substituting $\mathcal{L}(Q_{hp})$ back to (3.11) to obtain an equation for $\mathcal{L}(M_{ij})$ of form

$$(\delta_{ij}^{kl} - \mathcal{L}(X_i^h X_j^p) T_{hp}^{kl}) \mathcal{L}(M_{kl}) = \mathcal{L}(X_i^k X_j^l) \mathcal{L}(P_{kl}), \quad (3.14)$$

Then equation (3.14) is a linear equation of $\mathcal{L}(M_{kl})$. Thus, the determinant of the coefficients, denoted by

$$H(\lambda) = \det [(\delta_{ij}^{kl} - \mathcal{L}(X_i^h X_j^p) T_{hp}^{kl})], \quad (3.15)$$

is the desired characteristic function.

We note that (3.14) contains N^2 linear equations. Nevertheless, we can simplify the calculation due to symmetry. For example, since $\mathcal{L}(M_{kl}) = \mathcal{L}(M_{lk})$ and $\mathcal{L}(P_{kl}) = \mathcal{L}(P_{lk})$ in (3.14), we only need to solve equations for $\mathcal{L}(M_{kl})$ with $k \leq l$, and therefore have $N(N+1)/2$ equations.

The above procedure gives a general framework to obtain the characteristic function. However, it is too complicate to obtain an explicit expression for general cases. In the next section, we study a 2-dimensional equation with a specific form.

Denote the matrices

$$M = (M_i^j(t))_{N \times N}, \quad N = (N_i^j(t))_{N \times N}, \quad F = (F_i^j(t))_{N \times N}, \quad Q = (Q_i^j(t))_{N \times N}.$$

Before introducing the results for the 2-dimensional equation, we give some estimates for $F(t)$, $M(t)$ and $N(t)$ for general situation. These estimations ensure the existence of Laplace transforms of $F(t)$, $M(t)$ and $N(t)$.

Lemma 3.5. *Let α_0 be defined at (2.8) and assume $\alpha_0 < 0$. Then for any $\alpha \in (\alpha_0, 0)$, there exists a positive constant $K_1 = K_1(\alpha, \phi)$ ($\phi \in C([-1, 0], \mathbb{R}^N)$) such that*

$$\|F(t)\| \leq K_1(1 - e^{2\alpha t}), \quad t > 0. \quad (3.16)$$

Proof. From Theorem 3.3, for any $\alpha \in (\alpha_0, 0)$, there exists a positive constant $\tilde{K} = \tilde{K}(\alpha)$ such that

$$\begin{aligned} E(\Sigma_l^k(s)) &< \|\mu\| + \tilde{K} \sum_{j=1}^2 \left(|\sigma_l^{jk}| + e^{-\alpha} |\eta_l^{jk}| \right) \|\phi\| e^{\alpha s} \\ &< \|\mu\| + \tilde{K}(\|\sigma\| + e^{-\alpha} \|\eta\|) \|\phi\| e^{\alpha s} \end{aligned}$$

for any k, l . Here $\|\cdot\|$, as we mentioned before, mean the L^1 norm of a tensor. Hence,

$$|P_p(s)| \leq \left(\|\mu\| + \tilde{K}(\|\sigma\| + e^{-\alpha}\|\eta\|)\|\phi\|e^{\alpha s} \right)^2.$$

From Theorem 2.1, there exists a positive constant $\bar{K} = \bar{K}(\alpha)$ such that $\|X(t)\| < \bar{K}e^{\alpha t}$. Therefore, from (3.8), it is not difficult to have a constant K_1 (defined by \tilde{K} and \bar{K}) so that (3.16) holds.

Lemma 3.6. *Let α_0 be defined at (2.8) and assume $\alpha_0 < 0$. For any $\alpha \in (\alpha_0, 0)$, there exists $\nu = \nu(\alpha)$ such that*

$$\|M(t)\| \leq K_1 e^{\nu t}, \quad t \geq 0, \quad (3.17)$$

and

$$\|N(t)\| \leq (1 + e^{-\nu}) K_1 e^{\nu t}, \quad t \geq 0, \quad (3.18)$$

where K_1 is defined as in Lemma 3.5.

Proof. From the Cauchy-Schwarz inequality, we obtain for $i, j = 1, \dots, N$,

$$|N_{ij}(t)| = |E(\tilde{x}_i(t)\tilde{x}_j(t-1))| \leq \frac{M_{ii}(t) + M_{jj}(t-1)}{2}. \quad (3.19)$$

Then

$$\|N(t)\| \leq \frac{1}{2} \sum_{i,j=1}^N (M_{ii}(t) + M_{jj}(t-1)) \leq \|M(t)\| + \|M(t-1)\|. \quad (3.20)$$

Hence, from (3.7) and (3.20), there exists $C_0 > 0$ so that

$$\begin{aligned} |Q_{lp}(s)| &\leq \left| \sigma_l^{jk} \delta_{k,m} \sigma_p^{qm} M_{jq}(s) \right| + \left| \eta_l^{jk} \delta_{k,m} \sigma_p^{qm} M_{jq}(s-1) \right| \\ &\quad + \left| (\sigma_l^{jk} \delta_{k,m} \eta_p^{qm} + \eta_l^{qk} \delta_{k,m} \sigma_p^{jm}) N_{jq}(t) \right| \\ &\leq C_0 (\|M(s)\| + \|M(s-1)\|). \end{aligned} \quad (3.21)$$

Thus, from Lemma 3.5, we obtain for any $\alpha \in (\alpha_0, 0)$,

$$\begin{aligned}
\|M(t)\| &\leq \|F(t)\| + \int_0^t \|X(t-s)\|^2 \|Q(s)\| ds \\
&\leq K_1(1 - e^{2\alpha t}) + C_0 \bar{K}^2 \int_0^t e^{2\alpha(t-s)} (\|M(s)\| + \|M(s-1)\|) ds \\
&\leq K_1 + C_0 \bar{K}^2 \int_0^t e^{2\alpha(t-s)} \|M(s)\| ds + C_0 \bar{K}^2 e^{-2\alpha} \int_{-1}^{t-1} e^{2\alpha(t-s)} \|M(s)\| ds \\
&\leq K_1 + C_0 \bar{K}^2 (1 + e^{-2\alpha}) \int_0^t \|M(s)\| ds.
\end{aligned}$$

Applying the Gronwall inequality, we have

$$\|M(t)\| \leq K_1 e^{\nu t},$$

where $\nu = C_0 \bar{K}^2 (1 + e^{-2\alpha})$. The estimation (3.18) is obtained from (3.17) and (3.20).

4. Application to 2-dimensional equations

In this section, we apply the general framework established in the above to a special case of a 2-dimensional equation that $N = K = 2$ and $\mu_i^k = \sigma_i^{jk} = \eta_i^{jk} = 0$ when $i \neq k$. Hereafter, we do not use the Einstein summation convention, and introduce following notations for simplicity: $\mu_i = \mu_i^i$, $\sigma_i^j = \sigma_i^{ji}$, $\eta_i^j = \eta_i^{ji}$ ($i, j = 1, 2$). Thus, the equation we studied becomes

$$\left\{ \begin{aligned} dx_1(t) &= \sum_{j=1}^2 (a_1^j x_j(t) + b_1^j x_j(t-1)) dt \\ &\quad + \left(\mu_1 + \sum_{j=1}^2 (\sigma_1^j x_j(t) + \eta_1^j x_j(t-1)) \right) dW_1, \\ dx_2(t) &= \sum_{j=1}^2 (a_2^j x_j(t) + b_2^j x_j(t-1)) dt \\ &\quad + \left(\mu_2 + \sum_{j=1}^2 (\sigma_2^j x_j(t) + \eta_2^j x_j(t-1)) \right) dW_2, \end{aligned} \right. \quad (4.1)$$

In this particular case, the expressions of $P_{ij}(t)$, $Q_{ij}(t)$ and $F_{ij}(t)$, $M_{ij}(t)$, $N_{ij}(t)$ ($i, j = 1, 2$) in the previous section are as follows:

$$\begin{aligned} P_{12}(t) &= P_{21}(t) = Q_{12}(t) = Q_{21}(t) = 0, \\ P_{ii}(t) &= \left(\mu_i + \sum_{j=1}^2 (\sigma_i^j Ex_j(t) + \eta_i^j Ex_j(t-1)) \right)^2 \geq 0, \\ Q_{ii}(t) &= E \left(\sum_{j=1}^2 (\sigma_i^j \tilde{x}_j(t) + \eta_i^j \tilde{x}_j(t-1)) \right)^2 \geq 0, \end{aligned}$$

and

$$\begin{aligned} F_{ij}(t) &= \sum_{k=1}^2 \int_0^t X_i^k(t-s) X_j^k(t-s) P_{kk}(s) ds, \\ M_{ij}(t) &= \sum_{k=1}^2 \int_0^t X_i^k(t-s) X_j^k(t-s) (P_{kk}(s) + Q_{kk}(s)) ds, \\ N_{ij}(t) &= \sum_{k=1}^2 \int_0^{t-1} X_i^k(t-s) X_j^k(t-1-s) (P_{kk}(s) + Q_{kk}(s)) ds. \end{aligned}$$

Before we state and prove the main results, we introduce some preliminaries below.

When $N = 2$, we consider the delay differential equation

$$\begin{cases} \frac{dx_1}{dt} = \sum_{j=1}^2 (a_1^j x_j(t) + b_1^j x_j(t-1)), \\ \frac{dx_2}{dt} = \sum_{j=1}^2 (a_2^j x_j(t) + b_2^j x_j(t-1)). \end{cases} \quad (4.2)$$

The characteristic function of (4.2) is given by

$$\begin{aligned} h(\lambda) &= \det(\Delta(\lambda)) = \det \begin{pmatrix} \lambda - a_1^1 - b_1^1 e^{-\lambda} & -a_1^2 - b_1^2 e^{-\lambda} \\ -a_2^1 - b_2^1 e^{-\lambda} & \lambda - a_2^2 - b_2^2 e^{-\lambda} \end{pmatrix} \\ &= \lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda} + re^{-2\lambda}, \end{aligned} \quad (4.3)$$

where

$$a = -a_1^1 - a_2^2, \quad b = a_1^1 a_2^2 - a_1^2 a_2^1, \quad c = -b_1^1 - b_2^2,$$

$$d = a_1^1 b_2^2 + a_2^2 b_1^1 - a_1^2 b_2^1 - a_2^1 b_1^2, \quad r = b_1^1 b_2^2 - b_1^2 b_2^1.$$

The Laplace transform of the fundamental matrix is

$$\mathcal{L}(X)(\lambda) = \Delta^{-1}(\lambda) = \frac{1}{h(\lambda)} \begin{pmatrix} \lambda - a_2^2 - b_2^2 e^{-\lambda} & a_1^2 + b_1^2 e^{-\lambda} \\ a_2^1 + b_2^1 e^{-\lambda} & \lambda - a_1^1 - b_1^1 e^{-\lambda} \end{pmatrix}. \quad (4.4)$$

Here we give some properties of the fundamental solution $X(t)$ that are useful for the estimate of the second moment below.

Recall

$$\alpha_0 = \sup\{\operatorname{Re}(\lambda) : h(\lambda) = 0, \lambda \in \mathbb{C}\}. \quad (4.5)$$

When $\alpha_0 < 0$, from (4.4), we have

$$X(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{-iT}^{iT} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \Delta^{-1}(i\omega) d\omega.$$

Thus, from (4.4), we obtain

$$\begin{aligned} X_1^1(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega, \\ X_1^2(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_1^2 + b_1^2 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega, \\ X_2^1(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_2^1 + b_2^1 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega, \\ X_2^2(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_1^1 - b_1^1 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega. \end{aligned}$$

Obviously, $X_i^k(t)X_j^k(t)$ and $X_i^k(t)X_j^k(t-1)$ ($i, j, k = 1, 2$) have Laplace transforms. When $\alpha_0 < 0$, explicit expressions and estimates for the Laplace transforms $\mathcal{L}(X_i^k(t)X_j^k(t))$, $\mathcal{L}(X_i^k(t)X_j^k(t-1))$ are given in Appendix B.

4.1. Unboundedness of the second moment

Here we give a result for the unboundedness of the second moment of (4.1) when $\alpha_0 > 0$. First, we note a rare situation when the coefficients $\mu_i, \sigma_i^j, \eta_i^j$ ($i, j = 1, 2$) satisfy the following assumption.

Assumption H: $\mu_1 = \mu_2 = 0$, and there is a root $\lambda \in \mathbb{R}$ of $h(\lambda) = 0$ and an eigenvector $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2$ corresponding to the eigenvalue λ such that

$$\sum_{j=1}^2 (\sigma_i^j + e^{-\lambda} \eta_i^j) c_j = 0, \quad i = 1, 2.$$

When the assumption H is satisfied, the stochastic equation (4.1) has a deterministic solution $x(t) = e^{\lambda t} c$ ($t \geq 0$) with initial function $\phi = e^{\lambda t} c$ ($-1 \leq t < 0$), and the corresponding second moment $M(t) = 0$. This is a very rare situation and is excluded in discussions below.

The following result shows that in general the second moment of (4.1) is unbounded when the trivial solution of (2.3) is unstable.

Theorem 4.1. *Let α_0 be defined as (4.5) and $\alpha_0 > 0$. If the assumption H is not satisfied, the second moment of (4.1) is unbounded.*

Proof. We only need to show that there is a special solution $x(t; \phi)$ of (4.1) such that the corresponding second moment is unbounded. Note that $P_{ii}, Q_{ii} \geq 0$ ($i = 1, 2$), and hence

$$\begin{aligned} \|M(t)\| &\geq M_{11}(t) \geq F_{11}(t) \\ &= \sum_{i=1}^2 \int_0^t X_1^i(t-s)^2 P_{ii}(s) ds \geq 0. \end{aligned} \quad (4.6)$$

Let λ be a solution of $h(\lambda) = 0$ with $0 < \operatorname{Re}(\lambda) \leq \alpha_0$, and $c \in \mathbb{R}^2$ the corresponding eigenvector. Then $x_{\phi_1}(t) = \operatorname{Re}(e^{\lambda t} c)$ is the solution of (4.2) with initial function $\phi_1 = \operatorname{Re}(e^{\lambda t} c)$ ($-1 \leq t < 0$). Since the assumption H is not satisfied, $x_{\phi_1}(t)$ is not a solution of (4.1). Following the proof of Theorem 2.1 (3), there is a subset $U \subset \mathbb{R}^+$ with $m(U) = +\infty$ and positive constants C_1, C_2 such that for any $t \in U$,

$$|X_1^i(t)| \geq e^{\bar{\alpha} t}, \quad \bar{\alpha} \in (0, \alpha_0), \quad i = 1, 2$$

and

$$P_{11}(t) \geq C_1 \quad \text{or} \quad P_{22}(t) \geq C_2.$$

Thus, from (4.6), the second moment $M(t)$ is unbounded. \square

4.2. Boundedness of the second moments

Now, we always assume $\alpha_0 < 0$. Then (3.11)-(3.13) become

$$\mathcal{L}(M_{ij}) = \sum_{k=1}^2 \mathcal{L}(X_i^k X_j^k) (\mathcal{L}(P_{kk}) + \mathcal{L}(Q_{kk})), \quad (4.7)$$

$$\mathcal{L}(N_{ij}) = \sum_{k=1}^2 \mathcal{L}(X_i^k(t) X_j^k(t-1)) (\mathcal{L}(P_{kk}) + \mathcal{L}(Q_{kk})), \quad (4.8)$$

and

$$\begin{aligned} \mathcal{L}(Q_{ii}) &= \sum_{k=1}^2 ((\sigma_i^k)^2 + (\eta_i^k)^2 e^{-\lambda}) \mathcal{L}(M_{kk}) + 2 \sum_{k,l=1}^2 \sigma_i^k \eta_i^l \mathcal{L}(N_{kl}) \\ &\quad + 2 (\sigma_i^1 \sigma_i^2 + \eta_i^1 \eta_i^2 e^{-\lambda}) \mathcal{L}(M_{12}). \end{aligned} \quad (4.9)$$

Proposition 4.2. *Define the matrices*

$$A(\lambda) = \begin{pmatrix} \mathcal{L}((X_1^1)^2) & \mathcal{L}((X_1^2)^2) \\ \mathcal{L}((X_2^1)^2) & \mathcal{L}((X_2^2)^2) \end{pmatrix}, \quad G(\lambda) = \begin{pmatrix} G_1^1(\lambda) & G_1^2(\lambda) \\ G_2^1(\lambda) & G_2^2(\lambda) \end{pmatrix} \quad (4.10)$$

with

$$G_k^q(\lambda) = (\sigma_k^q)^2 + (\eta_k^q)^2 e^{-\lambda} + \sum_{p=1}^2 T_k^p S_p^q \quad (k, q = 1, 2), \quad (4.11)$$

$$T_k^p = 2(\sigma_k^1 \sigma_k^2 + \eta_k^1 \eta_k^2 e^{-\lambda}) \mathcal{L}(X_1^p X_2^p) + 2 \sum_{m,l=1}^2 \sigma_k^m \eta_k^l \mathcal{L}(X_m^p(t) X_l^p(t-1)) \quad (4.12)$$

and $S = (S_p^q)_{2 \times 2} = A(\lambda)^{-1}$ for $\text{Re}(\lambda) > \alpha_A$, where

$$\alpha_A \triangleq \sup \{ \text{Re}(\lambda) : \det(A(\lambda)) = 0, \lambda \in \mathbb{C} \}.$$

Then

$$\mathcal{L}(M_{12}) = \sum_{k,i=1}^2 \mathcal{L}(X_1^k X_2^k) S_k^i \mathcal{L}(M_{ii}) \quad (4.13)$$

and

$$\begin{pmatrix} \mathcal{L}(M_{11}) \\ \mathcal{L}(M_{22}) \end{pmatrix} = (I - D(\lambda))^{-1} A(\lambda) \begin{pmatrix} \mathcal{L}(P_{11}) \\ \mathcal{L}(P_{22}) \end{pmatrix}, \quad (4.14)$$

where $D(\lambda) = A(\lambda)G(\lambda)$.

Proof. Assume $\text{Re}(\lambda) > \alpha_A$ and therefore $S = A(\lambda)^{-1}$ is well defined. From (4.7), we have

$$\mathcal{L}(M_{ii}) = \sum_{k=1}^2 \mathcal{L}((X_i^k)^2)(\mathcal{L}(P_{kk}) + \mathcal{L}(Q_{kk})), \quad i = 1, 2,$$

therefore

$$\mathcal{L}(P_{kk}) + \mathcal{L}(Q_{kk}) = \sum_{i=1}^2 S_k^i \mathcal{L}(M_{ii}), \quad k = 1, 2, \quad (4.15)$$

which gives (4.13) from (4.7).

From (4.8) and (4.15),

$$\mathcal{L}(N_{ij}) = \sum_{p,q=1}^2 S_p^q \mathcal{L}(X_i^p(t) X_j^p(t-1)) \mathcal{L}(M_{qq}).$$

Thus, from (4.9), (4.10)-(4.12) and (4.13), we obtain

$$\begin{aligned} \mathcal{L}(Q_{ii}) &= \sum_{q=1}^2 \left[((\sigma_i^q)^2 + (\eta_i^q)^2 e^{-\lambda}) + 2 \sum_{m,l,p=1}^2 \sigma_i^m \eta_i^l S_p^q \mathcal{L}(X_m^p(t) X_l^p(t-1)) \right] \mathcal{L}(M_{qq}) \\ &\quad + 2(\sigma_i^1 \sigma_i^2 + \eta_i^1 \eta_i^2 e^{-\lambda}) \sum_{p=1}^2 \mathcal{L}(X_1^p X_2^p) \sum_{q=1}^2 S_p^q \mathcal{L}(M_{qq}) \\ &= \sum_{q=1}^2 \left[((\sigma_i^q)^2 + (\eta_i^q)^2 e^{-\lambda}) + \sum_{p=1}^2 S_p^q \left(2(\sigma_i^1 \sigma_i^2 + \eta_i^1 \eta_i^2 e^{-\lambda}) \mathcal{L}(X_1^p X_2^p) \right. \right. \\ &\quad \left. \left. + 2 \sum_{m,l=1}^2 \sigma_i^m \eta_i^l \mathcal{L}(X_m^p(t) X_l^p(t-1)) \right) \right] \mathcal{L}(M_{qq}) \\ &= \sum_{q=1}^2 G_i^q \mathcal{L}(M_{qq}). \end{aligned} \quad (4.16)$$

Hence, from (4.7) and (4.16), we have

$$\mathcal{L}(M_{ii}) = \sum_{k=1}^2 \mathcal{L}(X_i^k X_i^k) \mathcal{L}(P_{kk}) + \sum_{k=1}^2 \mathcal{L}(X_i^k X_i^k) \sum_{q=1}^2 G_k^q \mathcal{L}(M_{qq}),$$

i.e.,

$$(I - D(\lambda)) \begin{pmatrix} \mathcal{L}(M_{11}) \\ \mathcal{L}(M_{22}) \end{pmatrix} = A(\lambda) \begin{pmatrix} \mathcal{L}(P_{11}) \\ \mathcal{L}(P_{22}) \end{pmatrix},$$

which yields (4.14), and the Proposition is proved. \square

Denote

$$H(\lambda) = \det(I - D(\lambda)). \quad (4.17)$$

From (4.14),

$$\begin{pmatrix} \mathcal{L}(M_{11}) \\ \mathcal{L}(M_{22}) \end{pmatrix} = \frac{\text{adj}(I - D(\lambda))}{H(\lambda)} A(\lambda) \begin{pmatrix} \mathcal{L}(P_{11}) \\ \mathcal{L}(P_{22}) \end{pmatrix}, \quad (4.18)$$

where $\text{adj}(\cdot)$ denotes the adjoint matrix.

Let

$$\bar{\alpha}_0 = \sup\{\text{Re}(\lambda) : h(\lambda) \det(A(\lambda)) = 0, \lambda \in \mathbb{C}\}. \quad (4.19)$$

Then $\bar{\alpha}_0 = \max\{\alpha_0, \alpha_A\}$ ³, and $A(\lambda)$ is invertible for $\text{Re}(\lambda) > \bar{\alpha}_0$. Thus $D(\lambda)$ and $\frac{\text{adj}(I - D(\lambda))}{H(\lambda)}$ are analytic for $\text{Re}(\lambda) > \bar{\alpha}_0$.

In the following theorem, we show that $H(\lambda)$ is the characteristic function for the second moment boundedness.

Theorem 4.3. *Let $H(\lambda) = \det(I - D(\lambda))$ and $D(\lambda)$ be defined as in Proposition 4.2. Let $\bar{\alpha}_0$ be defined at (4.19) and assume $\bar{\alpha}_0 < 0$. Then*

- (i) *if all roots of the equation $H(\lambda) = 0$ have negative real parts, then the second moment for any solution of (4.1) is bounded, and $M(t)$ approaches a 2×2 constant matrix exponentially as $t \rightarrow +\infty$;*
- (ii) *if the equation $H(\lambda) = 0$ has a root with positive real part, and the assumption H is not satisfied, then there exists a solution of (4.1) whose second moment is unbounded.*

To prove Theorem 4.3, we first give some useful Lemmas.

Lemma 4.4. *Let $D(\lambda)$ be defined as in Proposition 4.2 and assume $\bar{\alpha}_0 < 0$. Then there exist constants d_0 and T_0 such that for $|\lambda| \geq T_0$ and $\text{Re}(\lambda) > \bar{\alpha}_0$,*

$$\|D(\lambda)\| \leq \frac{d_0}{|\lambda|}. \quad (4.20)$$

³We conjecture that $\bar{\alpha}_0 = \alpha_0$, but are not able to prove.

Proof. From Lemma Appendix B.1, there exists R_0 so that $\|A(\lambda)\| < \frac{R_0}{|\lambda|}$ when $|\lambda|$ is large enough. Thus, it is enough to show that there exists a constant g_0 such that when $\operatorname{Re}(\lambda) > \bar{\alpha}_0$,

$$\limsup_{|\lambda| \rightarrow +\infty} |G_k^q(\lambda)| < g_0, \quad \forall k, q = 1, 2, \quad (4.21)$$

Then (4.20) is satisfied with $d_0 = R_0 g_0$.

To prove (4.21), we only need to show that when $\operatorname{Re}(\lambda) > \bar{\alpha}_0$,

$$\begin{aligned} \limsup_{|\lambda| \rightarrow +\infty} |S_p^q \mathcal{L}(X_1^p X_2^p)| &< +\infty, \\ \limsup_{|\lambda| \rightarrow +\infty} |S_p^q \mathcal{L}(X_m^p(t) X_l^p(t-1))| &< +\infty. \end{aligned} \quad (4.22)$$

Here we only give the proof of the first result of (4.22) for $p = q = 1$, and the others are similar.

For any $\operatorname{Re}(\lambda) > \bar{\alpha}_0$, from Lemma Appendix B.1, we obtain

$$S_1^1(\lambda) = \frac{\mathcal{L}((X_2^2)^2)}{\det(A(\lambda))} = \frac{\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{22}(\lambda)}{\det(A(\lambda))},$$

where

$$\begin{aligned} \det(A(\lambda)) &= \mathcal{L}((X_1^1)^2)(\lambda) \mathcal{L}((X_2^2)^2)(\lambda) - \mathcal{L}((X_1^2)^2)(\lambda) \mathcal{L}((X_2^1)^2)(\lambda) \\ &= [\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{11}(\lambda)] \\ &\quad \times [\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{22}(\lambda)] \\ &\quad - [a_1^2 + b_1^2 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{12}(\lambda)] [a_2^1 + b_2^1 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{21}(\lambda)]. \end{aligned}$$

Thus, from (B.7), when $\operatorname{Re}(\lambda) > \bar{\alpha}_0$, $\det(A(\lambda)) \neq 0$ and there exists a constant $s_{11} > 0$ such that

$$\lim_{|\lambda| \rightarrow +\infty} |\lambda S_1^1(\lambda)| = \lim_{|\lambda| \rightarrow +\infty} \frac{|\lambda (\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0) g_{22}(\lambda))|}{|\det(A(\lambda))|} = s_{11}$$

Hence, from (B.8), there exist two positive constants \tilde{T} and $\tilde{R}_{11} = 2s_{11}$ such that for $|\lambda| > \tilde{T}$ and $\operatorname{Re}(\lambda) > \bar{\alpha}_0$,

$$|\lambda^2 S_1^1 \mathcal{L}(X_1^1 X_2^1)| \leq \tilde{R}_{11}, \quad (4.23)$$

which implies

$$\limsup_{|\lambda| \rightarrow \infty} |S_1^1 \mathcal{L}(X_1^1 X_1^2)| < +\infty,$$

and the first inequality in (4.22) for $p = q = 1$ is proved. \square

From Lemma 4.4, when $\operatorname{Re}(\lambda) > \bar{\alpha}_0$, $H(\lambda)$ is analytic and

$$\lim_{|\lambda| \rightarrow +\infty} |H(\lambda)| = 1.$$

Thus there is a real number β_0 such that all roots of $H(\lambda) = 0$ satisfy $\operatorname{Re}(\lambda) < \beta_0$ (refer to the discussion in [5, Lemma 4.1 in Chapter 1]), where

$$\beta_0 = \sup\{\operatorname{Re}(\lambda) : H(\lambda) = 0, \lambda \in \mathbb{C}\}. \quad (4.24)$$

Let

$$Z(t) = \mathcal{L}^{-1} \left(\frac{1}{H(\lambda)} \operatorname{adj}(I - D(\lambda)) A(\lambda) \right), \quad (4.25)$$

then from (4.18),

$$\begin{pmatrix} M_{11}(t) \\ M_{22}(t) \end{pmatrix} = \int_0^t Z(t-s) \begin{pmatrix} P_{11}(s) \\ P_{22}(s) \end{pmatrix} ds. \quad (4.26)$$

The following result gives an estimation of $Z(t)$.

Lemma 4.5. *Let β_0 be defined at (4.24) and assume $\bar{\alpha}_0 < 0$. There exists a positive constant C_3 such that for any $\beta > \max\{\bar{\alpha}_0, \beta_0\}$,*

$$\|Z(t)\| < C_3 e^{\beta t}. \quad (4.27)$$

The proof is similar to that of [5, Theorem 5.2 in Chapter 1] and the details are given in Appendix C.

Lemma 4.6. *Assume $\bar{\alpha}_0 < 0$. If $M_{11}(t), M_{22}(t)$ are bounded, then $M_{12}(t)$ is also bounded for any $t > 0$.*

Proof. Assume that there exists a positive constant M_0 so that

$$M_{ii}(t) \leq M_0, \quad \forall t > 0, \quad i = 1, 2.$$

Let

$$Y_i(t) = \mathcal{L}^{-1} \left(\sum_{k=1}^2 \mathcal{L}(X_1^k X_2^k) S_k^i \right), \quad i = 1, 2.$$

Then (4.13) yields

$$M_{12}(t) = \sum_{i=1}^2 \int_0^t Y_i(t-s) M_{ii}(s) ds. \quad (4.28)$$

Similar to the proof of (4.23), there exist positive constants \tilde{R}_{ik} ($i, k = 1, 2$) and T^* such that for $|\lambda| > T^*$ and $\text{Re}(\lambda) > \bar{\alpha}_0$,

$$|\lambda^2 \mathcal{L}(X_1^k X_2^k) S_k^i| \leq \tilde{R}_{ik}.$$

Thus, when $\bar{\alpha}_0 < 0$, similar to the proof of Lemma 4.5, there exist positive constants ρ_i ($i = 1, 2$) such that for any $\alpha > \bar{\alpha}_0$,

$$\|Y_i(t)\| \leq \rho_i e^{\alpha t}, \quad i = 1, 2. \quad (4.29)$$

Hence, from (4.29), when $\bar{\alpha}_0 < 0$, for any $\alpha \in (\bar{\alpha}_0, 0)$,

$$\begin{aligned} |M_{12}(t)| &\leq M_0 \int_0^t (|Y_1(s)| + |Y_2(s)|) ds \leq (\rho_1 + \rho_2) M_0 \int_0^t e^{\alpha s} ds \\ &\leq \frac{2(\rho_1 + \rho_2) M_0}{|\alpha|}, \end{aligned}$$

i.e., $M_{12}(t)$ is bounded. \square

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $H(\lambda) = \det(I - D(\lambda))$ and $D(\lambda)$ be defined as in Proposition 4.2, and β_0 be defined as (4.24). We also assume $\bar{\alpha}_0 < 0$.

From (3.2), for any $\alpha \in (\bar{\alpha}_0, 0)$, there exists a constant $K_2 = K_2(\phi)$ ($\phi \in C([-1, 0], \mathbb{R}^N)$) such that

$$P_{ii}(t) \leq (\mu_i)^2 + K_2 e^{\alpha t}, \quad \forall t \geq 0, i = 1, 2. \quad (4.30)$$

The prove is straightforward from

$$\begin{aligned} P_{ii}(t) &= (\mu_i + \sigma_i^1 Ex_1(t) + \eta_i^1 Ex_1(t-1) + \sigma_i^2 Ex_2(t) + \eta_i^2 Ex_2(t-1))^2 \\ &\leq \left(|\mu_i| + (|\sigma_i^1| + |\sigma_i^2|) \|Ex(t)\| + (|\eta_i^1| + |\eta_i^2|) \|Ex(t-1)\| \right)^2 \\ &\leq (\mu_i)^2 + e^{\alpha t} \left[\tilde{K}^2 \|\phi\|^2 (|\sigma_i^1| + |\sigma_i^2|)^2 + \tilde{K}^2 \|\phi\|^2 e^{-2\alpha} (|\eta_i^1| + |\eta_i^2|)^2 \right. \\ &\quad \left. + 2\tilde{K} \|\phi\| |\mu_i| (|\sigma_i^1| + |\sigma_i^2|) + 2\tilde{K} \|\phi\| e^{-\alpha} |\mu_i| (|\eta_i^1| + |\eta_i^2|) \right. \\ &\quad \left. + 2\tilde{K}^2 \|\phi\|^2 e^{-\alpha} (|\sigma_i^1| + |\sigma_i^2|) (|\eta_i^1| + |\eta_i^2|) \right]. \end{aligned}$$

(i). Assume $\beta_0 < 0$. Let

$$\hat{M}(t) = \begin{pmatrix} M_{11}(t) \\ M_{22}(t) \end{pmatrix}, \quad \mu_0 = (\mu_1)^2 + (\mu_2)^2.$$

From Lemma 4.5, for any $\beta \in (\max\{\bar{\alpha}_0, \beta_0\}, 0)$, $\alpha \in (\bar{\alpha}_0, 0)$ and $\alpha \neq \beta$, from (4.27),

$$\begin{aligned} \|\hat{M}\| &\leq \int_0^t \|Z(s)\| (P_{11}(t-s) + P_{22}(t-s)) ds \\ &\leq C_3 \int_0^t e^{\beta s} (\mu_0 + 2K_2 e^{\alpha(t-s)}) ds \\ &= C_3 \left(\mu_0 \frac{1 - e^{\beta t}}{-\beta} + 2K_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \right) \\ &\leq C_3 \left(\frac{\mu_0}{|\beta|} + \frac{4K_2}{|\alpha - \beta|} \right). \end{aligned}$$

From Lemma 4.6, the second moment $M(t)$ is bounded for any initial function $\phi \in C([-1, 0], \mathbb{R}^2)$.

Let

$$\hat{M}_\infty = \begin{pmatrix} \hat{M}_{\infty,1} \\ \hat{M}_{\infty,2} \end{pmatrix} = \int_0^{+\infty} Z(s) \begin{pmatrix} (\mu_1)^2 \\ (\mu_2)^2 \end{pmatrix} ds.$$

It is easy to have

$$\|\hat{M}_\infty\| \leq \mu_0 \int_0^{+\infty} \|Z(s)\| ds \leq C_3 \mu_0 \int_0^{+\infty} e^{\beta s} ds = \frac{C_3 \mu_0}{|\beta|}. \quad (4.31)$$

Thus, from (4.27),

$$\begin{aligned} \|\hat{M}(t) - \hat{M}_\infty\| &= \left\| \int_0^t Z(s) \begin{pmatrix} P_{11}(t-s) - (\mu_1)^2 \\ P_{22}(t-s) - (\mu_2)^2 \end{pmatrix} ds - \int_t^{+\infty} Z(s) \begin{pmatrix} (\mu_1)^2 \\ (\mu_2)^2 \end{pmatrix} ds \right\| \\ &\leq 2K_2 \int_0^t \|Z(s)\| e^{\alpha(t-s)} ds + \mu_0 \int_t^{+\infty} \|Z(s)\| ds \\ &\leq C_3 \mu_0 \frac{e^{\beta t}}{-\beta} + 2C_3 K_2 \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} \\ &\leq C_3 \left(\frac{\mu_0}{|\beta|} + \frac{4K_2}{|\alpha - \beta|} \right) e^{t \max\{\alpha, \beta\}} \triangleq C_4 e^{t \max\{\alpha, \beta\}}, \end{aligned} \quad (4.32)$$

which implies $\|\hat{M}(t) - \hat{M}_\infty\| \rightarrow 0$ (as $t \rightarrow +\infty$) exponentially since $\max\{\alpha, \beta\} < 0$.

From (4.28), and let

$$M_\infty = \begin{pmatrix} \hat{M}_{\infty,1} & \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds \\ \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds & \hat{M}_{\infty,2} \end{pmatrix}.$$

Obviously, from (4.29) and (4.31), M_∞ is bounded. Thus

$$\begin{aligned} \|M(t) - M_\infty\| &= \left\| \begin{pmatrix} M_{11} - \hat{M}_{\infty,1} & M_{12} - \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds \\ M_{12} - \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds & M_{22} - \hat{M}_{\infty,2} \end{pmatrix} \right\| \\ &= \|\hat{M}(t) - \hat{M}_\infty\| + 2 \left| M_{12} - \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds \right|. \end{aligned}$$

From (4.28), (4.29), (4.31) and (4.32), we obtain for any $\tilde{\alpha} \in (\bar{\alpha}_0, 0)$ and $\tilde{\alpha} \neq \max\{\alpha, \beta\}$,

$$\begin{aligned} & \left| M_{12} - \sum_{i=1}^2 \int_0^{+\infty} Y_i(s) \hat{M}_{\infty,i} ds \right| \\ & \leq \sum_{i=1}^2 \left(\int_0^t |Y_i(s)| |M_{ii}(t-s) - \hat{M}_{\infty,i}| ds + \int_t^{+\infty} |Y_i(s)| |\hat{M}_{\infty,i}| ds \right) \\ & \leq C_4(\rho_1 + \rho_2) \int_0^t e^{\tilde{\alpha}s} e^{(t-s)\max\{\alpha, \beta\}} ds + \frac{C_3\mu_0(\rho_1 + \rho_2)}{|\beta|} \int_t^{+\infty} e^{\tilde{\alpha}s} ds \\ & = C_4(\rho_1 + \rho_2) \frac{e^{\tilde{\alpha}t} - e^{t\max\{\alpha, \beta\}}}{\tilde{\alpha} - \max\{\alpha, \beta\}} + \frac{C_3\mu_0(\rho_1 + \rho_2)}{|\tilde{\alpha}\beta|} e^{\tilde{\alpha}t} \\ & \leq \left(\frac{2C_4(\rho_1 + \rho_2)}{|\tilde{\alpha} - \max\{\alpha, \beta\}|} + \frac{C_3\mu_0(\rho_1 + \rho_2)}{|\tilde{\alpha}\beta|} \right) e^{t\max\{\tilde{\alpha}, \alpha, \beta\}}. \end{aligned}$$

Therefore

$$\|M(t) - M_\infty\| \leq \left(C_4 + \frac{4C_4(\rho_1 + \rho_2)}{|\tilde{\alpha} - \max\{\alpha, \beta\}|} + \frac{2C_3\mu_0(\rho_1 + \rho_2)}{|\tilde{\alpha}\beta|} \right) e^{t\max\{\tilde{\alpha}, \alpha, \beta\}},$$

which implies that $M(t)$ approaches to M_∞ exponentially as $t \rightarrow +\infty$.

(ii). Now we assume $\beta_0 > 0$. We only need to show that there is a special solution $x(t; \phi)$ such that the corresponding second moment is unbounded. Similar to the proof of Theorem 4.1, let $\lambda = \alpha + i\omega$ ($\alpha \leq \alpha_0 < 0$) be a solution of $h(\lambda) = 0$, and $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in \mathbb{R}^2$ is an eigenvector corresponding to the eigenvalue λ , then $x_{\phi_2}(t) = \text{Re}(e^{\lambda t} c)$ ($t \geq -1$) is a solution of (2.3) with initial function $\phi_2 = \text{Re}(e^{\lambda t} c) \in C([-1, 0], \mathbb{R}^2)$. Hence, for this particular initial function ϕ_2 , since the assumption H is not satisfied, $x_{\phi_2}(t)$ is not a solution of (1.1) and therefore $P_{11}(t)$ or $P_{22}(t)$ is nonzero. Thus the Laplacian $\mathcal{L}(P_{11})$ or $\mathcal{L}(P_{22})$ is nonzero.

Since $\|M(t)\| \geq \|\hat{M}(t)\| \geq M_{11}(t)$, in the following, we only need to show that $M_{11}(t)$ is unbounded for the initial function ϕ_2 . From (4.18), we have

$$M_{11}(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{\bar{c}-iT}^{\bar{c}+iT} e^{st} \left[\frac{(1-d_{22})\mathcal{L}((X_1^1)^2)}{H(s)} \mathcal{L}(P_{11})(s) + \frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(s)} \mathcal{L}(P_{22})(s) \right] ds,$$

where $\bar{c} > \beta_0$. Here $\frac{(1-d_{22})\mathcal{L}((X_1^1)^2)}{H(s)}$ and $\frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(s)}$ are analytic functions for $\text{Re}(s) > \bar{\alpha}_0$ and are nonzeros. It is easy to see that $\mathcal{L}(P_{11})(s)$, $\mathcal{L}(P_{22})(s)$ are analytic for $\text{Re}(s) = \bar{c} > 0$. Thus, similar to the proof of Theorem 2.1 (3), there is a sequence $\{t_k\}_{k \geq 1}$ such that $t_k \rightarrow +\infty$ and $M_{11}(t_k) \rightarrow +\infty$ as $k \rightarrow +\infty$, which implies that the second moment is unbounded. Thus, the theorem is proved. \square

The critical case of $\bar{\alpha}_0 < 0$, $\beta_0 = 0$ is not considered here, and the issue of boundedness criteria remains open.

The characteristic function $H(\lambda)$ depends not only on the coefficients of equation (4.1), but also on the Laplace transforms of $X_i^k(t)X_j^k(t)$ and $X_i^k(t)X_j^k(t-1)$ ($i, j, k = 1, 2$). One can calculate these functions numerically according to Lemma Appendix B.1 and Appendix B.2, however, it is difficult to obtain $\beta_0 = \sup\{\text{Re}(\lambda) : H(\lambda) = 0\}$ for a given equation. Hence the sufficient conditions for the second moment boundedness of (4.1) established in Theorem 4.3 are not practical. In applications, one need to derive useful criteria in terms of equation coefficients, either from the proposed characteristic function or following the procedure in the above discussions. Here, for applications, we give a practical condition for the boundedness of the second moment.

Theorem 4.7. Assume $\alpha_0 < 0$. If there exists $\alpha \in (\alpha_0, 0)$ and the a positive constant $\bar{K} = \bar{K}(\alpha)$ so that

$$\|X(t)\| \leq \bar{K}e^{-\alpha t}, \quad t > 0$$

and

$$\left(|\sigma_1^1| + |\eta_1^1| + |\sigma_1^2| + |\eta_1^2|\right)^2 + \left(|\sigma_2^1| + |\eta_2^1| + |\sigma_2^2| + |\eta_2^2|\right)^2 < -\frac{\alpha}{2\bar{K}^2}, \quad (4.33)$$

then the second moment $M(t)$ is bounded.

Proof. From the expression of $Q_{ii}(t)$ and (3.19), we obtain for $i = 1, 2$,

$$\begin{aligned} Q_{ii}(t) &\leq \left(|\sigma_i^1| + |\sigma_i^2|\right)\left(|\sigma_i^1| + |\eta_i^1| + |\sigma_i^2| + |\eta_i^2|\right)\left[M_{11}(t) + 2|M_{12}(t)| + M_{22}(t)\right] \\ &\quad + \left(|\eta_i^1| + |\eta_i^2|\right)\left(|\sigma_i^1| + |\eta_i^1| + |\sigma_i^2| + |\eta_i^2|\right)\left[M_{11}(t-1) \right. \\ &\quad \left. + 2|M_{12}(t-1)| + M_{22}(t-1)\right] \\ &= \left(|\sigma_i^1| + |\sigma_i^2|\right)\left(|\sigma_i^1| + |\eta_i^1| + |\sigma_i^2| + |\eta_i^2|\right)\|M(t)\| \\ &\quad + \left(|\eta_i^1| + |\eta_i^2|\right)\left(|\sigma_i^1| + |\eta_i^1| + |\sigma_i^2| + |\eta_i^2|\right)\|M(t-1)\|. \end{aligned}$$

Since $Q_{ii}(t) \geq 0$, for any $\alpha \in (\alpha_0, 0)$ and $i, j = 1, 2$,

$$\begin{aligned} |M_{ij}(t)| &\leq |F_{ij}(t)| + \sum_{k=1}^2 \int_0^t |X_i^k(t-s)X_j^k(t-s)Q_{kk}(s)|ds \\ &\leq |F_{ij}(t)| + \alpha_2 \bar{K}^2 e^{2\alpha t} \int_0^t e^{-2\alpha s} \|M(s)\| ds \\ &\quad + \alpha_3 \bar{K}^2 e^{2\alpha t} \int_0^t e^{-2\alpha s} \|M(s-1)\| ds, \end{aligned}$$

where

$$\begin{aligned} \alpha_2 &= \left(|\sigma_1^1| + |\sigma_1^2|\right)\left(|\sigma_1^1| + |\eta_1^1| + |\sigma_1^2| + |\eta_1^2|\right) \\ &\quad + \left(|\sigma_2^1| + |\sigma_2^2|\right)\left(|\sigma_2^1| + |\eta_2^1| + |\sigma_2^2| + |\eta_2^2|\right), \\ \alpha_3 &= \left(|\eta_1^1| + |\eta_1^2|\right)\left(|\sigma_1^1| + |\eta_1^1| + |\sigma_1^2| + |\eta_1^2|\right) \\ &\quad + \left(|\eta_2^1| + |\eta_2^2|\right)\left(|\sigma_2^1| + |\eta_2^1| + |\sigma_2^2| + |\eta_2^2|\right). \end{aligned}$$

Thus from Lemma 3.5, we have

$$\begin{aligned} \|M(t)\| = \sum_{i,j=1}^2 |M_{ij}| &\leq K_1(1 - e^{2\alpha t}) + 4\alpha_2 \bar{K}^2 e^{2\alpha t} \int_0^t e^{-2\alpha s} \|M(s)\| ds \\ &\quad + 4\alpha_3 \bar{K}^2 e^{2\alpha t} \int_0^t e^{-2\alpha s} \|M(s-1)\| ds. \end{aligned}$$

Let

$$y(t) = e^{-2\alpha t} \|M(t)\|, \quad r(t) = K_1(e^{-2\alpha t} - 1)$$

and

$$\bar{p} = 4\alpha_2 \bar{K}^2, \quad \bar{q} = 4\alpha_3 \bar{K}^2 e^{-2\alpha}.$$

Then

$$y(t) \leq \bar{p} \int_0^t y(s) ds + \bar{q} \int_0^t y(s-1) ds + r(t), \quad t \geq 0.$$

Choose $\lambda = -2\alpha > 0$, we get

$$\sup_{t \geq 0} |r(t)e^{-\lambda t}| = \sup_{t \geq 0} K_1(1 - e^{2\alpha t}) \leq 2K_1 < +\infty.$$

By (4.33), we obtain

$$\lambda - \bar{p} - \bar{q}e^{-\lambda} = -2\alpha - 4(\alpha_2 + \alpha_3)\bar{K}^2 > -2\alpha + 4\bar{K}^2 \frac{\alpha}{2\bar{K}^2} = 0.$$

Therefore from Lemma 3.9 in [8], there exists $C_5 = C_5(\alpha)$ such that

$$\|M(t)\|e^{-2\alpha t} = y(t) \leq C_5 e^{-2\alpha t}, \quad t \geq 0,$$

that is, $\|M(t)\| \leq C_5$ for all $t \geq 0$. \square

In this paper, we have established framework procedure to calculate the characteristic function for the second moment boundedness of linear delay differential equations with a single discrete delay, we also applied the procedure to study a special case of 2-dimensional equations. However, as we have seen, the resulting function has a very complicate form. These complicate results is in fact show the elaborate correlations of non-Markov processes when both delay and stochastic effects are taken into involved. In spite of the complicate form of final formulations, the procedure of calculating is simple and easy to follow. Thus, in applications, one can develop the characteristic function, for particular equation of studied, following the scheme given here. We leave these further applications to future works.

Appendix A. Proof of Theorem 2.1 (iii)

Proof. Let $\alpha_1 < \alpha_0$. Since the zeros of $h(\lambda)$ are isolated, we can take $\bar{\alpha} \in (\alpha_1, \alpha_0)$ such that $\text{Re}(\lambda) = \bar{\alpha}$ does not contain any root of $h(\lambda) = 0$. Next, choose $c_1 > \alpha_0$ and $T > 0$, then

$$X(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{c_1 - iT}^{c_1 + iT} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda. \quad (\text{A.1})$$

To calculate the integral (A.1), we consider the integration of the matrix $e^{\lambda t} \Delta^{-1}(\lambda)$ around the bounder of the box in the complex plane with boundary $\gamma = \gamma_1 \gamma_2 \gamma_3 \gamma_4$ in the anticlockwise direction, where the segment γ_1 is the set $\{c_1 + i\tau : -T \leq \tau \leq T\}$, the segment γ_3 is the set $\{\bar{\alpha} + i\tau : -T \leq \tau \leq T\}$, the segment γ_2 is the set $\{u + iT : \bar{\alpha} \leq u \leq c_1\}$ and the segment γ_4 is the set $\{u - iT : \bar{\alpha} \leq u \leq c_1\}$. Then Cauchy theorem of residues implies

$$\oint_{\gamma} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda = 2\pi i \sum_{j=1}^m \text{Res}_{\lambda=\lambda_j} e^{\lambda t} \Delta^{-1}(\lambda) \neq 0,$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are roots of $h(\lambda) = 0$ inside γ ($m \geq 1$ from the definition of α_0 , and $m < +\infty$ since $h(\lambda)$ is an analytic function). We also assume that

$$\bar{\alpha} < \text{Re}(\lambda_1) \leq \text{Re}(\lambda_2) \leq \dots \leq \text{Re}(\lambda_m) = \alpha_0.$$

Note that

$$\text{Res}_{\lambda=\lambda_j} e^{\lambda t} \Delta^{-1}(\lambda) = P_j(t) e^{\lambda_j t},$$

where $P_j(t) = (P_j^{kl}(t))_{N \times N}$ with $P_j^{kl}(t)$ ($k, l = 1, \dots, N$) a polynomial of t with degree given by the multiplicity of λ_j minus 1. Thus,

$$\oint_{\gamma} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda = 2\pi i \sum_{j=1}^m P_j(t) e^{\lambda_j t}. \quad (\text{A.2})$$

From the definition of the adjoint matrix, we obtain

$$\begin{aligned} \left\| \int_{\gamma_2} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda \right\| &= \left\| - \int_{\bar{\alpha} + iT}^{c_1 + iT} e^{\lambda t} \frac{\text{adj}(\Delta(\lambda))}{h(\lambda)} d\lambda \right\| \\ &\leq e^{c_1 t} \int_{\bar{\alpha}}^{c_1} \left\| \frac{\text{adj}(\Delta(u + iT))}{h(u + iT)} \right\| du \rightarrow 0 \text{ (as } T \rightarrow +\infty), \end{aligned}$$

where $\text{adj}(\Delta(\lambda))$ is the adjoint matrix of $\Delta(\lambda)$. In the same way, we have

$$\left\| \int_{\gamma_4} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda \right\| \rightarrow 0 \text{ (as } T \rightarrow +\infty).$$

Therefore by (A.2) we get

$$\frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{c_1 - iT}^{c_1 + iT} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda + \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{\bar{\alpha} + iT}^{\bar{\alpha} - iT} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda = \sum_{j=1}^m P_j(t) e^{\lambda_j t},$$

i.e.,

$$X(t) = X_{\bar{\alpha}}(t) + \sum_{j=1}^m P_j(t)e^{\lambda_j t},$$

where

$$X_{\bar{\alpha}}(t) = \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{\bar{\alpha}-iT}^{\bar{\alpha}+iT} e^{\lambda t} \Delta^{-1}(\lambda) d\lambda \triangleq (X_{\bar{\alpha}}^{kl}(t))_{N \times N}.$$

Moreover, similar to the proof of Theorem 2.1 (i) (refer [8] or [17]), there exists a positive constant $\bar{C} = \bar{C}(\bar{\alpha})$ such that $X_{\bar{\alpha}}(t)$ satisfies

$$\|X_{\bar{\alpha}}(t)\| \leq \bar{C}e^{\bar{\alpha}t}, \quad t \geq 0.$$

Thus we obtain for and $k, l = 1, \dots, N$

$$\begin{aligned} |X_k^l(t)| &\geq \left| \sum_{j=1}^m P_j^{kl}(t)e^{\lambda_j t} \right| - |X_{\bar{\alpha}}^{kl}(t)| \geq \left| \sum_{j=1}^m P_j^{kl}(t)e^{\lambda_j t} \right| - \bar{C}e^{\bar{\alpha}t} \\ &= e^{\bar{\alpha}t} (e^{(\operatorname{Re}(\lambda_1) - \bar{\alpha})t} f(t) - \bar{C}), \end{aligned}$$

where $f(t) = \left| \sum_{j=1}^m P_j^{kl}(t)e^{(\lambda_j - \operatorname{Re}(\lambda_1))t} \right|$.

Let $\lambda_j = \beta_j + i\omega_j$ ($j = 1, 2, \dots, m$), and assume n_0 such that $\beta_j < \beta_m$ when $1 \leq j \leq n_0$ and $\beta_j = \beta_m$ when $n_0 + 1 \leq j \leq m$. Then

$$\begin{aligned} f(t) &= e^{(\beta_m - \beta_1)t} \left| \sum_{j=1}^{n_0} e^{-(\beta_m - \beta_j)t} P_j^{kl}(t)e^{i\omega_j t} + \sum_{j=n_0+1}^m P_j^{kl}(t)e^{i\omega_j t} \right| \\ &\geq \left| \sum_{j=n_0+1}^m \operatorname{Re}(P_j^{kl}(t)e^{i\omega_j t}) \right| - \sum_{j=1}^{n_0} e^{-(\beta_m - \beta_j)t} |P_j^{kl}(t)|. \end{aligned}$$

Since $P_j^{kl}(t)$ ($j = 1, 2, \dots, m$) are nonzero polynomials, we find that

$$\begin{aligned} \sum_{j=n_0+1}^m \operatorname{Re}(P_j^{kl}(t)e^{i\omega_j t}) &= t^{n_1} \left[\sum_{j=n_0+1}^m a_j \cos(\omega_j t) + b_j \sin(\omega_j t) + O(t^{-1}) \right] \\ &= t^{n_1} \left[\sum_{j=n_0+1}^m \sqrt{a_j^2 + b_j^2} \sin(\theta_j + \omega_j t) + O(t^{-1}) \right] \end{aligned}$$

(as $t \rightarrow +\infty$), where a_j, b_j ($j = n_0 + 1, \dots, m$) are constants and $a_j^2 + b_j^2 \neq 0$, n_1 is the highest degree of the polynomials $P_j^{kl}(t)$ ($j = n_0 + 1, \dots, m$), and

$$\sin(\theta_j) = \frac{a_j}{\sqrt{a_j^2 + b_j^2}}, \quad \cos(\theta_j) = \frac{b_j}{\sqrt{a_j^2 + b_j^2}} \quad (j = n_0 + 1, \dots, m).$$

Thus, we can always find a subset $U_0 \subset \mathbb{R}^+$ with measure $m(U_0) = +\infty$ such that all functions

$$\sin(\theta_j + \omega_j t) > \varepsilon, \quad t \in U_0, \quad n_0 + 1 \leq j \leq m$$

for some small positive constant ε , and therefore the subset U is always possible by taking $U = U_0 \cap (t_0, +\infty)$ with t_0 large enough. Hence for the above ε and $\forall t \in U$,

$$\left| \sum_{j=n_0+1}^m \operatorname{Re} (P_j^{kl}(t) e^{i\omega_j t}) \right| > 2\varepsilon.$$

Furthermore, since $\sum_{j=1}^{n_0} e^{-(\beta_m - \beta_j)t} |P_j^{kl}(t)| \rightarrow 0$ as $t \rightarrow +\infty$, we can take U such that

$$e^{(\operatorname{Re}(\lambda_1) - \bar{\alpha})t} f(t) - \bar{C} > 1, \quad \forall t \in U$$

and hence for any $k, l = 1, \dots, N$ and $t \in U$,

$$|X_k^l(t)| \geq e^{\bar{\alpha}t} (e^{(\operatorname{Re}(\lambda_1) - \bar{\alpha})t} f(t) - \bar{C}) > e^{\bar{\alpha}t},$$

therefore (2.9) is concluded. □

Appendix B. Expressions and estimates of the Laplace transforms

$$\mathcal{L}\left(X_i^k(t)X_j^k(t)\right) \text{ and } \mathcal{L}\left(X_i^k(t)X_j^k(t-1)\right)$$

Lemma Appendix B.1. *Assume $\alpha_0 < 0$. For any $\text{Re}(\lambda) > \alpha_0$ ($\lambda \in \mathbb{C}$),*

$$\begin{aligned} \mathcal{L}((X_1^1)^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} e^{i\omega t} d\omega \\ &= \frac{\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{11}(\lambda), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} \mathcal{L}((X_1^2)^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_1^2 + b_1^2 e^{-i\omega}}{h(i\omega)} \cdot \frac{a_1^2 + b_1^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\ &= \frac{a_1^2 + b_1^2 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{12}(\lambda), \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} \mathcal{L}((X_2^1)^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_2^1 + b_2^1 e^{-i\omega}}{h(i\omega)} \cdot \frac{a_2^1 + b_2^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\ &= \frac{a_2^1 + b_2^1 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{21}(\lambda), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \mathcal{L}((X_2^2)^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_1^1 - b_1^1 e^{-i\omega}}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_1^1 - b_1^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\ &= \frac{\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{22}(\lambda), \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} \mathcal{L}(X_1^1 X_2^1)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_2^1 + b_2^1 e^{-i\omega}}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\ &= \frac{a_2^1 + b_2^1 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{31}(\lambda), \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \mathcal{L}(X_1^2 X_2^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_1^2 + b_1^2 e^{-i\omega}}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_1^1 - b_1^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\ &= \frac{a_1^2 + b_1^2 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{32}(\lambda), \end{aligned} \quad (\text{B.6})$$

where

$$\begin{aligned}
g_{11}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(a_2^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_2^2 - [(c + b_2^2)(\lambda - i\omega) - \alpha_0 b_2^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
g_{12}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_1^2 + b_1^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_1^2 - a)(\lambda - i\omega) - b - \alpha_0 a_1^2 - [(c - b_1^2)(\lambda - i\omega) + \alpha_0 b_1^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
g_{21}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_2^2 + b_2^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_2^2 - a)(\lambda - i\omega) - b - \alpha_0 a_2^2 - [(c - b_2^2)(\lambda - i\omega) + \alpha_0 b_2^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
g_{22}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_1^2 - b_1^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(a_1^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_1^2 - [(c + b_1^2)(\lambda - i\omega) - \alpha_0 b_1^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
g_{31}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_2^2 + b_2^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(a_2^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_2^2 - [(c + b_2^2)(\lambda - i\omega) - \alpha_0 b_2^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
g_{32}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a_1^2 + b_1^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(a_1^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_1^2 - [(c + b_1^2)(\lambda - i\omega) - \alpha_0 b_1^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega.
\end{aligned}$$

Furthermore, there exist positive constants c_{ij} ($i = 1, 2, 3, j = 1, 2$) and R_{ij} ($i, j = 1, 2$) such that for $\text{Re}(\lambda) > \alpha_0$,

$$\begin{aligned}
\lim_{|\lambda| \rightarrow +\infty} |g_{ij}(\lambda)| &= 0 \quad (i = 1, 2, 3, j = 1, 2), \\
\lim_{|\lambda| \rightarrow +\infty} |\lambda g_{12}(\lambda)| &= c_{12}, \quad \lim_{|\lambda| \rightarrow +\infty} |h(\lambda - \alpha_0) g_{ii}(\lambda)| = c_{ii} \quad (i = 1, 2), \\
\lim_{|\lambda| \rightarrow +\infty} |\lambda g_{21}(\lambda)| &= c_{21}, \quad \lim_{|\lambda| \rightarrow +\infty} |h(\lambda - \alpha_0) g_{3j}(\lambda)| = c_{3j} \quad (j = 1, 2)
\end{aligned} \tag{B.7}$$

and

$$\begin{aligned}
\lim_{|\lambda| \rightarrow +\infty} |\lambda \mathcal{L}((X_i^j)^2)(\lambda)| &= R_{ij} \quad (i, j = 1, 2), \\
\lim_{|\lambda| \rightarrow +\infty} |\lambda \mathcal{L}(X_1^1 X_2^1)(\lambda)| &= 0, \quad \lim_{|\lambda| \rightarrow +\infty} |\lambda \mathcal{L}(X_1^2 X_2^2)(\lambda)| = 0.
\end{aligned} \tag{B.8}$$

Proof. For any $\text{Re}(\lambda) > 2\alpha_0$ ($\lambda \in \mathbb{C}$),

$$\begin{aligned}
\mathcal{L}((X_1^1)^2)(\lambda) &= \int_0^{+\infty} e^{-\lambda t} X_1^1(t) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \int_0^{+\infty} e^{-(\lambda-i\omega)t} X_1^1(t) dt d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega.
\end{aligned}$$

Since $\int_0^{+\infty} e^{-(\lambda-i\omega-\alpha_0)t} dt = \frac{1}{\lambda - i\omega - \alpha_0}$ for $\text{Re}(\lambda) > \alpha_0$, then

$$\begin{aligned}
\mathcal{L}((X_1^1)^2)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \frac{1}{\lambda - i\omega - \alpha_0} d\omega \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \left[\frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} - \frac{1}{\lambda - i\omega - \alpha_0} \right] d\omega \\
&= \int_0^{+\infty} e^{-(\lambda-\alpha_0)t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} e^{i\omega t} d\omega dt + g_{11}(\lambda) \\
&= \int_0^{+\infty} e^{-(\lambda-\alpha_0)t} X_1^1(t) dt + g_{11}(\lambda) \\
&= \frac{\lambda - \alpha_0 - a^2 - b_2^2 e^{-(\lambda-\alpha_0)}}{h(\lambda - \alpha_0)} + g_{11}(\lambda)
\end{aligned}$$

for $\text{Re}(\lambda) > \alpha_0$, where

$$\begin{aligned}
g_{11}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{i\omega - a_2^2 - b_2^2 e^{-i\omega}}{h(i\omega)} \times \\
&\quad \frac{-(a_2^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_2^2 - [(c + b_2^2)(\lambda - i\omega) - \alpha_0 b_2^2 + d] e^{-(\lambda-i\omega)} - r e^{-2(\lambda-i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega.
\end{aligned}$$

Note that $g_{11}(\lambda)$ is convergent for any $\text{Re}(\lambda) > \alpha_0$. Hence there exist two positive constants c_{11} and R_{11} such that when $\text{Re}(\lambda) > \alpha_0$,

$$\lim_{|\lambda| \rightarrow +\infty} |g_{11}(\lambda)| = 0, \quad \lim_{|\lambda| \rightarrow +\infty} |h(\lambda - \alpha_0)g_{11}(\lambda)| = c_{11}, \quad \lim_{|\lambda| \rightarrow +\infty} |\lambda \mathcal{L}((X_1^1)^2)(\lambda)| = R_{11}.$$

Other expressions in (B.1)-(B.6), (B.7) and (B.8) can be obtained similarly.

□

Similar to Lemma Appendix B.1, we have the following expressions and estimates.

Lemma Appendix B.2. Assume $\alpha_0 < 0$. For any $\text{Re}(\lambda) > \alpha_0$ ($\lambda \in \mathbb{C}$),

$$\begin{aligned}
\mathcal{L}(X_1^1(t)X_1^1(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_2^2 - b_2^2 e^{-i\omega})}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{11}(\lambda), \\
\mathcal{L}(X_1^2(t)X_1^2(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_1^2 + b_1^2 e^{-i\omega})}{h(i\omega)} \cdot \frac{a_1^2 + b_1^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (a_1^2 + b_1^2 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{12}(\lambda), \\
\mathcal{L}(X_2^1(t)X_2^1(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_2^1 + b_2^1 e^{-i\omega})}{h(i\omega)} \cdot \frac{a_2^1 + b_2^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (a_2^1 + b_2^1 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{21}(\lambda), \\
\mathcal{L}(X_2^2(t)X_2^2(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_1^1 - b_1^1 e^{-i\omega})}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_1^1 - b_1^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{22}(\lambda), \\
\mathcal{L}(X_1^1(t)X_2^1(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_2^1 + b_2^1 e^{-i\omega})}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_2^2 - b_2^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (a_2^1 + b_2^1 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{31}(\lambda), \\
\mathcal{L}(X_2^1(t)X_1^1(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_2^2 - b_2^2 e^{-i\omega})}{h(i\omega)} \cdot \frac{a_2^1 + b_2^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{32}(\lambda), \\
\mathcal{L}(X_1^2(t)X_2^2(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_1^1 - b_1^1 e^{-i\omega})}{h(i\omega)} \cdot \frac{a_1^2 + b_1^2 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (\lambda - \alpha_0 - a_1^1 - b_1^1 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{41}(\lambda), \\
\mathcal{L}(X_2^2(t)X_1^2(t-1))(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_1^2 + b_1^2 e^{-i\omega})}{h(i\omega)} \cdot \frac{\lambda - i\omega - a_1^1 - b_1^1 e^{-(\lambda-i\omega)}}{h(\lambda - i\omega)} d\omega \\
&= \frac{e^{-(\lambda-\alpha_0)} (a_1^2 + b_1^2 e^{-(\lambda-\alpha_0)})}{h(\lambda - \alpha_0)} + \tilde{g}_{42}(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{g}_{11}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_2^2 - b_2^2 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(a_2^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_2^2 - [(c + b_2^2)(\lambda - i\omega) - \alpha_0 b_2^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{12}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_1^2 + b_1^2 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_1^2 - a)(\lambda - i\omega) - b - \alpha_0 a_1^2 - [(c - b_1^2)(\lambda - i\omega) + \alpha_0 b_1^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{21}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_2^1 + b_2^1 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_2^1 - a)(\lambda - i\omega) - b - \alpha_0 a_2^1 - [(c - b_2^1)(\lambda - i\omega) + \alpha_0 b_2^1 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{22}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_1^1 - b_1^1 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(a_1^1 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_1^1 - [(c + b_1^1)(\lambda - i\omega) - \alpha_0 b_1^1 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{31}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_2^1 + b_2^1 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(a_2^2 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_2^2 - [(c + b_2^2)(\lambda - i\omega) - \alpha_0 b_2^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{32}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_2^2 - b_2^2 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_2^1 - a)(\lambda - i\omega) - b - \alpha_0 a_2^1 - [(c - b_2^1)(\lambda - i\omega) + \alpha_0 b_2^1 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{41}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (i\omega - a_1^1 - b_1^1 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(\lambda - i\omega)^2 + (a_1^2 - a)(\lambda - i\omega) - b - \alpha_0 a_1^2 - [(c - b_1^2)(\lambda - i\omega) + \alpha_0 b_1^2 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega, \\
\tilde{g}_{42}(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega} (a_1^2 + b_1^2 e^{-i\omega})}{h(i\omega)} \times \\
&\quad \frac{-(a_1^1 + a + \alpha_0)(\lambda - i\omega) - b + \alpha_0 a_1^1 - [(c + b_1^1)(\lambda - i\omega) - \alpha_0 b_1^1 + d] e^{-(\lambda - i\omega)} - r e^{-2(\lambda - i\omega)}}{h(\lambda - i\omega)(\lambda - i\omega - \alpha_0)} d\omega.
\end{aligned}$$

Moreover, there exist positive constants \tilde{c}_{ij} ($i = 1, 2, 3, 4, j = 1, 2$) such that for $\text{Re}(\lambda) > \alpha_0$,

$$\lim_{|\lambda| \rightarrow +\infty} |\tilde{g}_{ij}(\lambda)| = 0 \quad (i = 1, 2, 3, 4, j = 1, 2), \quad (\text{B.9})$$

and when $i = 1, 3$

$$\lim_{|\lambda| \rightarrow +\infty} |h(\lambda - \alpha_0) \tilde{g}_{i1}(\lambda)| = \tilde{c}_{i1}, \quad \lim_{|\lambda| \rightarrow +\infty} |\lambda \tilde{g}_{i2}(\lambda)| = \tilde{c}_{i2},$$

when $i = 2, 4$,

$$\lim_{|\lambda| \rightarrow +\infty} |h(\lambda - \alpha_0) \tilde{g}_{i2}(\lambda)| = \tilde{c}_{i2}, \quad \lim_{|\lambda| \rightarrow +\infty} |\lambda \tilde{g}_{i1}(\lambda)| = \tilde{c}_{i1}. \quad (\text{B.10})$$

Appendix C. Proof of Lemma 4.5

Proof. Assume $\bar{\alpha}_0 < 0$. Let

$$Z(t) = \mathcal{L}^{-1} \left(\frac{1}{H(\lambda)} \text{adj}(I - D(\lambda))A(\lambda) \right) \triangleq (Z_{ij}(t))_{2 \times 2}.$$

We only prove that there exists a positive constant K_{11} such that for any $\beta > \max\{\bar{\alpha}_0, \beta_0\}$,

$$|Z_{11}(t)| \leq K_{11}e^{\beta t}, \quad t > 0.$$

The estimates of $Z_{12}(t)$, $Z_{ii}(t)$ ($i = 1, 2$) are similar and omitted.

Let

$$D(\lambda) = \begin{pmatrix} d_{11}(\lambda) & d_{12}(\lambda) \\ d_{21}(\lambda) & d_{22}(\lambda) \end{pmatrix},$$

then

$$Z_{11}(t) = \mathcal{L}^{-1} \left(\frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} \right) + \mathcal{L}^{-1} \left(\frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\lambda)} \right). \quad (\text{C.1})$$

Now, we estimate the two terms in (C.1) respectively.

When $\beta > \beta_0$, $H(\beta + i\omega) \neq 0$ for any $\omega \in \mathbb{R}$. Thus

$$\begin{aligned} \mathcal{L}^{-1} \left(\frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} \right) &= \int_{(c_2)} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} e^{\lambda t} d\lambda \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow +\infty} \int_{c_2 - iT}^{c_2 + iT} e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda, \end{aligned}$$

where $c_2 > \beta$ is large enough. First we want to prove that

$$\mathcal{L}^{-1} \left(\frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} \right) = \int_{(\beta)} e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda. \quad (\text{C.2})$$

To this end, we consider the integration of the function $e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)}$ around the boundary of the box Γ in the complex plan with the boundary $\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ in the anticlockwise direction, where the segment Γ_1 is the set $\{c_2 + i\tau : -T \leq \tau \leq T\}$, the segment Γ_2 is the set $\{u + iT : \beta \leq u \leq c_2\}$, the segment Γ_3 is the set $\{\beta + i\tau : -T \leq \tau \leq T\}$ and the segment Γ_4 is the set

$\{u - iT : \beta \leq u \leq c_2\}$. Since $H(\lambda)$ has no zeros in this box Γ , the integral over the boundary is zero, i.e.,

$$\left(\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} \right) \left(e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} \right) d\lambda = 0.$$

Thus, (C.2) is concluded if

$$\lim_{T \rightarrow +\infty} \int_{\Gamma_i} e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda = 0 \quad (i = 2, 4).$$

Since $H(\lambda) = (1 - d_{11})(1 - d_{22}) - d_{12}d_{21}$, we have

$$\frac{H(\lambda)}{1 - d_{22}} = 1 - d_{11} - \frac{d_{12}d_{21}}{1 - d_{22}}.$$

From Lemma 4.4, there exists a constant $T_1 \geq T_0 > 0$ such that when $|\lambda| \geq T_1$ and $\operatorname{Re}(\lambda) > \bar{\alpha}_0$,

$$\left| \frac{H(\lambda)}{1 - d_{22}} \right| \geq 1 - |d_{11}| - \frac{|d_{12}||d_{21}|}{1 - |d_{22}|} \geq 1 - \frac{d_0}{|\lambda|} - \frac{\frac{d_0^2}{|\lambda|^2}}{1 - \frac{d_0}{|\lambda|}} \geq \frac{1}{2}.$$

Hence from (B.8), for $|\lambda| \geq T_1$ and $\operatorname{Re}(\lambda) > \bar{\alpha}_0$, $|\mathcal{L}((X_1^1)^2)| \leq \frac{1 + R_{11}}{|\lambda|}$, and therefore

$$\begin{aligned} \left| \int_{\Gamma_2} e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda \right| &= \left| \int_{c_2 + iT}^{\beta + iT} e^{\lambda t} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda \right| \\ &\leq 2 \int_{\beta}^{c_2} e^{ut} \frac{1 + R_{11}}{\sqrt{u^2 + T^2}} du \\ &\leq \frac{2e^{c_2 t}(1 + R_{11})}{T} (c_2 - \beta) \rightarrow 0 \quad (\text{as } T \rightarrow +\infty). \end{aligned}$$

Similarly,

$$\int_{\Gamma_4} \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} e^{\lambda t} d\lambda \rightarrow 0 \quad (\text{as } T \rightarrow +\infty).$$

Thus, (C.2) is obtained.

Let T_1 as above, and

$$\begin{aligned} W(\lambda) &= \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} - \frac{1}{\lambda - \beta_0} \\ &= \frac{(1 - d_{22})\mathcal{L}((X_1^1)^2)}{H(\lambda)} \cdot \frac{\lambda - \beta_0 - \frac{H(\lambda)}{(1 - d_{22})\mathcal{L}((X_1^1)^2)}}{\lambda - \beta_0}. \end{aligned}$$

From Lemma Appendix B.1, we have for $\text{Re}(\lambda) > \bar{\alpha}_0$,

$$\begin{aligned} &\lambda - \beta_0 - \frac{H(\lambda)}{(1 - d_{22})\mathcal{L}((X_1^1)^2)} \\ &= \lambda - \frac{1}{\mathcal{L}((X_1^1)^2)} - \beta_0 - \frac{1}{\mathcal{L}((X_1^1)^2)} \left(\frac{H(\lambda)}{1 - d_{22}} - 1 \right) \\ &= -\beta_0 - \frac{1}{\mathcal{L}((X_1^1)^2)} \left(\frac{H(\lambda)}{1 - d_{22}} - 1 \right) + \frac{(\alpha_0 - a - a_2^2)\lambda + \alpha_0 a - b - h(\lambda - \alpha_0)g_{11}}{\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0)g_{11}} \\ &\quad - \frac{(b_2^2 \lambda + c(\lambda - \alpha_0) + d) e^{-(\lambda - \alpha_0)} + r e^{-2(\lambda - \alpha_0)}}{\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0)g_{11}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\mathcal{L}((X_1^1)^2)} \left(\frac{H(\lambda)}{1 - d_{22}} - 1 \right) &= \frac{h(\lambda - \alpha_0)}{\lambda - \alpha_0 - a_2^2 - b_2^2 e^{-(\lambda - \alpha_0)} + h(\lambda - \alpha_0)g_{11}} \times \\ &\quad \left(-d_{11} - \frac{d_{12}d_{21}}{1 - d_{22}} \right). \end{aligned}$$

From (4.20), there exists a positive constant \tilde{r}_{11} such that for $|\lambda| \geq T_1$ and $\text{Re}(\lambda) > \bar{\alpha}_0$,

$$\left| \frac{1}{\mathcal{L}((X_1^1)^2)} \left(\frac{H(\lambda)}{1 - d_{22}} - 1 \right) \right| \leq \tilde{r}_{11}$$

and

$$\left| \lambda - \beta_0 - \frac{H(\lambda)}{(1 - d_{22})\mathcal{L}((X_1^1)^2)} \right| \leq |\alpha_0 - a - a_2^2| + |\beta_0| + \tilde{r}_{11} = |\alpha_0 + a_1^1| + |\beta_0| + \tilde{r}_{11},$$

which, applying (B.8), implies

$$|W(\lambda)| \leq \frac{|\alpha_0 + a_1^1| + |\beta_0| + \tilde{r}_{11}}{|\lambda - \beta_0|} \cdot \frac{2(1 + R_{11})}{|\lambda|} \leq \frac{2(1 + R_{11}) (|\alpha_0 + a_1^1| + |\beta_0| + \tilde{r}_{11})}{|\lambda||\lambda - \beta_0|}.$$

Thus, for $\beta > \max\{\bar{\alpha}_0, \beta_0\}$, since $\lambda - \beta_0 \neq 0$ and $H(\lambda) \neq 0$ ($\text{Re}(\lambda) > \beta_0$),

$$\begin{aligned}
\left| \mathcal{L}^{-1} \left(\frac{(1 - d_{22}) \mathcal{L}((X_1^1)^2)}{H(\lambda)} \right) \right| &= \left| \int_{(\beta)} e^{\lambda t} \frac{(1 - d_{22}) \mathcal{L}((X_1^1)^2)}{H(\lambda)} d\lambda \right| \\
&= \left| \int_{(\beta)} e^{\lambda t} W(\lambda) d\lambda + \int_{(\beta)} \frac{e^{\lambda t}}{\lambda - \beta_0} d\lambda \right| \\
&\leq e^{\beta_0 t} + \left| \int_{(\beta)} e^{\lambda t} W(\lambda) d\lambda \right| \\
&\leq e^{\beta t} + e^{\beta t} \left(\int_{-T_1}^{T_1} |W(\beta + i\tau)| d\tau \right. \\
&\quad \left. + 2 \int_{T_1}^{+\infty} \frac{2(1 + R_{11}) (|\alpha_0 + a_1^1| + |\beta_0| + \tilde{r}_{11})}{\sqrt{\beta^2 + \tau^2} \sqrt{(\beta - \beta_0)^2 + \tau^2}} d\tau \right) \\
&\triangleq \tilde{K}_{11} e^{\beta t}, \quad t > 0,
\end{aligned}$$

where

$$\tilde{K}_{11} = 1 + \int_{-T_1}^{T_1} |W(\beta + i\tau)| d\tau + 2 \int_{T_1}^{+\infty} \frac{2(1 + R_{11}) (|\alpha_0 + a_1^1| + |\beta_0| + \tilde{r}_{11})}{\sqrt{\beta^2 + \tau^2} \sqrt{(\beta - \beta_0)^2 + \tau^2}} d\tau.$$

Now, we consider the second term in (C.1). Since

$$\frac{H(\lambda)}{d_{12}(\lambda)} = \frac{(1 - d_{11})(1 - d_{22})}{d_{12}} - d_{21},$$

from Lemma 4.4, there exists a constant $T_2 \geq T_0 > 0$ such that when $|\lambda| \geq T_2$ and $\text{Re}(\lambda) > \bar{\alpha}_0$,

$$\begin{aligned}
\left| \frac{H(\lambda)}{\lambda d_{12}(\lambda)} \right| &\geq \frac{1}{|\lambda|} \left[\frac{(1 - |d_{11}|)(1 - |d_{22}|)}{|d_{12}|} - |d_{21}| \right] \\
&\geq \frac{(1 - \frac{d_0}{|\lambda|})(1 - \frac{d_0}{|\lambda|})}{d_0} - \frac{d_0}{|\lambda|^2} \geq \frac{1}{2d_0}.
\end{aligned}$$

Thus for $|\lambda| \geq T_2$, $\text{Re}(\lambda) > \bar{\alpha}_0$ and $i = 2, 4$, from (B.8),

$$\begin{aligned}
\left| \int_{\Gamma_i} e^{\lambda t} \frac{d_{12} \mathcal{L}((X_2^1)^2)}{H(\lambda)} d\lambda \right| &\leq \int_{\beta}^{c_2} e^{ut} \frac{2d_0(1 + R_{21})}{u^2 + T^2} du \\
&\leq \frac{2e^{c_2 t} d_0(1 + R_{21})}{T^2} (c_2 - \beta) \rightarrow 0 \quad (\text{as } T \rightarrow +\infty).
\end{aligned}$$

Hence

$$\mathcal{L}^{-1} \left(\frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\lambda)} \right) = \int_{(\beta)} e^{\lambda t} \frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\lambda)} d\lambda.$$

Therefore, from (B.8), we obtain for $\beta > \max\{\bar{\alpha}_0, \beta_0\}$,

$$\begin{aligned} \left| \mathcal{L}^{-1} \left(\frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\lambda)} \right) \right| &\leq e^{\beta t} \left(\int_{-T_2}^{T_2} \left| \frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\beta + i\tau)} \right| d\tau + 2 \int_{T_2}^{+\infty} \frac{2d_0(1 + R_{21})}{\beta^2 + \tau^2} d\tau \right) \\ &\triangleq \bar{K}_{11} e^{\beta t}, \quad t > 0, \end{aligned}$$

where

$$\bar{K}_{11} = \int_{-T_2}^{T_2} \left| \frac{d_{12}\mathcal{L}((X_2^1)^2)}{H(\beta + i\tau)} \right| d\tau + 2 \int_{T_2}^{+\infty} \frac{2d_0(1 + R_{21})}{\beta^2 + \tau^2} d\tau.$$

Taking $K_{11} = \tilde{K}_{11} + \bar{K}_{11}$, (C.1) is concluded and the Lemma is proved. \square

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